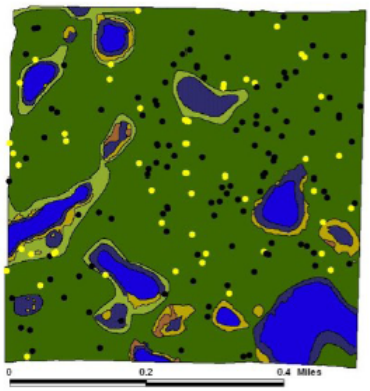
Spatial Point Patterns

- Observed locations of events: datum is the location
  - Major shift in interest from previous material!
  - Up to now, location has been fixed point or fixed area,
    - Location arbitrary or happenstance, often controlled by the investigator (where to take point samples)
    - Random quantity has been the value at each location
- Random quantity is now the location of an event
- May record additional information at each location ⇒ marked point process
  - Sometimes small # of classes
  - Examples: species of tree, live / dead plant, successful / unsuccessful bird nest, disease case / not diseased person
  - Or, may be continuous quantity
  - Examples: diameter of tree, angle of a crystal
- But that addn info only exists when there is an event at that location

Duck nests in a 1/4 section of ND



Potential questions

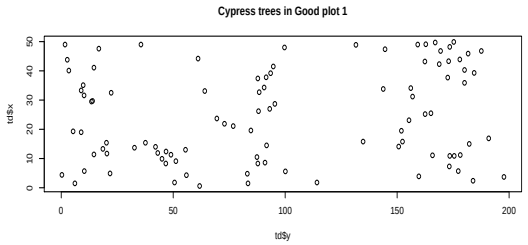
1. is intensity (# events / unit area) const. or vary. over the study area
2. how does intensity vary as function of potential covariates
   - EX: does intensity of duck nests decline with distance to wetland?
3. are events randomly scattered, clustered, or regular
   - EX: are duck nests independently located in space, or do they cluster near other duck nests, or do they avoid being near other nests?
4. how can we describe pattern at multiple scales?
5. how can we describe rel. between two (or more) types of points?
   - EX: do depredated nests tend to occur near other depredated nests?
6. how can we describe the cor. between marks as a function of distance?
   - when mark is a continuous value

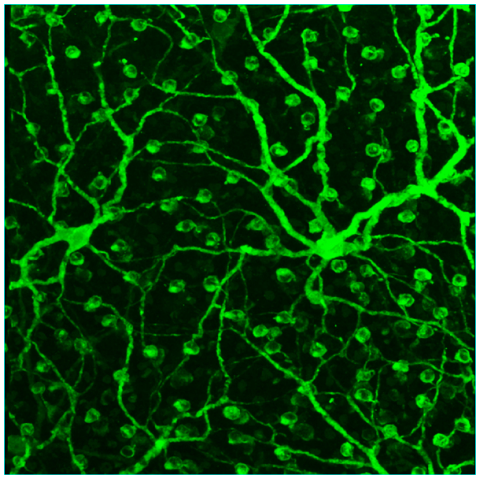
- Historically, 3) was most important Q
- Now, moving beyond to all the other Q.
- We'll begin with 3 and 4, then 5 and 6, end w/intensity

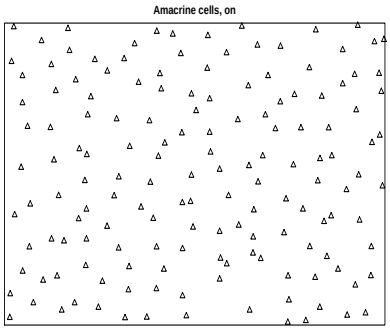
Application areas

- Many, including:
- Ecology: historically important field of application, many different applications, including:
  - spatial pattern (random / clustered / avoidance) of a single species
  - patterns of mortality (clustered or not?)
- transportation: locations of accidents
- neurology: locations of neurons
- geology: locations of earthquakes (space, or space/time)
- geography: do similar types of stores tend to cluster near each other?
- epidemiology: do cases of a particular disease cluster?
  - If so, suggests contagious disease or single spatial cause

Examples in pictures:







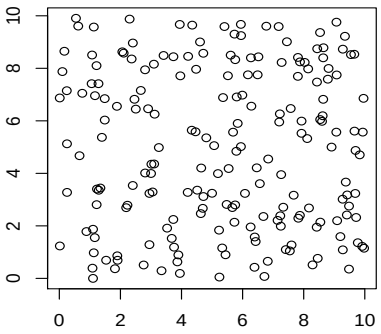
Homogeneous Poisson Process = Complete Spatial Randomness

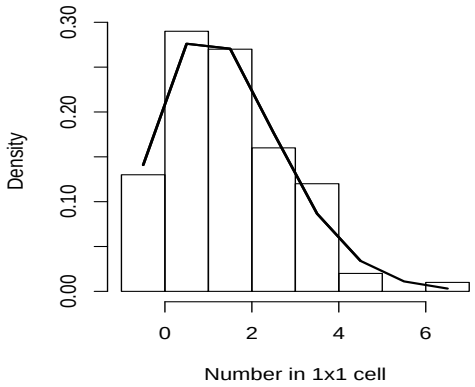
- imagine a very small area, dA, with P[event occurs in dA] $= \lambda dA$
- dA small enough that:
  - at most 1 event in dA
  - most areas have 0 events
- λ = expected # events / unit area
- λ is the intensity of the spatial process
- Two assumptions that give HPP = CSR
  - λ constant over study area
  - the outcome (0/1) in dA_1 is independent of the outcome in non-overlapping area dA_2

Homogeneous Poisson Processes

Some mathematical results:

- Define N_A = # points in area A (no longer small)
- $N_A \sim Poiss(\lambda\, A)$
  - mean #: $\lambda\, A$
  - var #: $\lambda\, A$
  - pmf $P[X \mid \lambda A] = \frac{e^{-\lambda A}(\lambda A)^X}{X!}$
- examples: CSR, observe 196 obs on (0,10), (0,10)
- Look at individual 1x1 quadrats
  - mean count per 1x1 quadrats = 1.96
  - Var count = 1.90
  - Histogram close to theoretical pmf

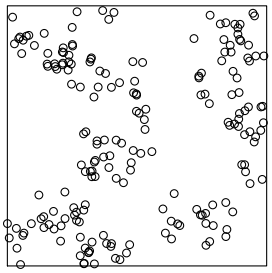


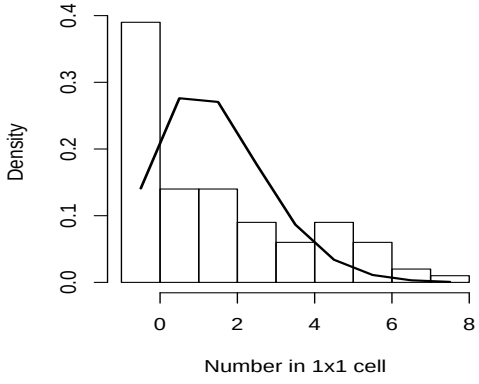


Clustering

- Clustered processes/pattern: points more likely to occur near other points.
- For quadrats, means that:
  - some quadrats contain a cluster, have more points than expected
  - other quadrats have no points
- same mean, larger variance
- for clustered process with 196 points on next two slides:
  - mean = 1.96, variance = 4.70

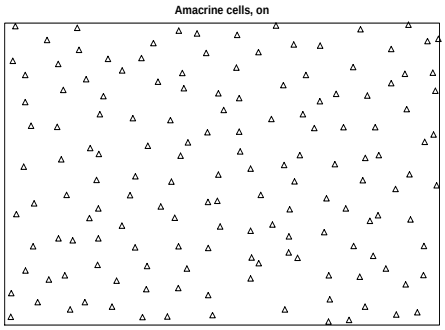
A clustered process





Regular processes

- P[event in dA] lower if dA close to another point
- Tends to "space out" points



A better approach

- Historically: Quadrats used extensively
- But, very limited. Restricted to one specific scale (size of quadrat)
- Better approach
  - record locations of events, not just count in a box
  - usually all events in a predefined area
  - can be random sample of events
  - But, hard to take a simple random sample
- Can convert to quadrat counts, but can do a LOT more with (x,y) data

Summary functions

- Concept:
  - measure something as a function of distance
- Various choices of summary
  - Distance to nearest neighbor (event - event distance)
  - Distance to nearest point (point - event distance)
  - Combination of these two
  - Ripley's K function
  - pair correlation function
- Each has uses

Distance to nearest neighbor

- How close is each obs. to its nearest neighbor?
  - clustering: NN distances tend to be small
  - random (CSR): intermediate
  - regular: NN distances tend to be large
- Historical: calculate mean NN distance, compare to theoretical value (Clark-Evans test)
- Current: estimate cdf of NN distance: $G(x) = P[\text{NN distance } \leq x]$
  - for each event: find NN, calculate distance to NN
  - hard part is finding NN. Some fancy and fast algorithms (see NN article)
  - compare estimated $\hat{G}(x)$ to theoretical $G(x)$ for CSR

Theoretical CDF on NN distance

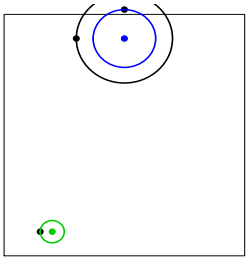
- $P[D < x] = 1 - e^{-\lambda \pi x^2}$
  - x is distance of concern,
  - λ is intensity (events per unit area)
  - πx^2 is area of circle, radius x
- (For the statisticians). Nice ex. of CDF method for deriving transforming a random variable
- Define D = distance to NN dist.

$$\begin{aligned} G(x) &= P[D \leq x] = 1 - P[D > x] \\ &= P[\text{no obs in circle of radius } x] \\ N_A &\sim \text{Poiss}(\lambda A), \text{so:} \\ P[0] &= \frac{e^{-\lambda A}(\lambda A)^0}{0!} \\ P[D \leq x] &= 1 - e^{-\lambda \pi x^2} \end{aligned}$$

- Similar ideas, different formula for 1D or 3D.

Edge effects

- Above assumes infinite plane
- Real study areas have edges
- When a point is close to edge of mapped area, what is the distance to the NN?

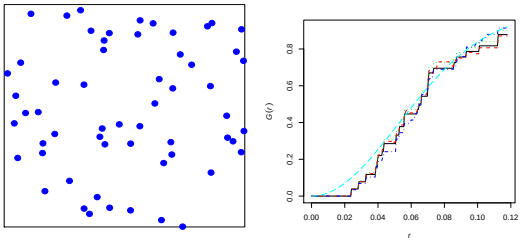


- overestimate D.
- true NN may be just over the boundary (close to event)
- observed NN (inside study area) is larger than it "should be"
- So underestimate $G(x)$ especially for large x

Edge effects

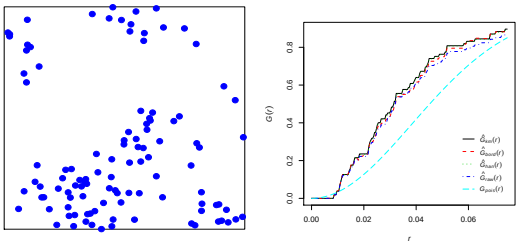
- Three approaches to edge effects
- 1) Ignore problem.
  - Study area edge is a real edge (e.g. lake shoreline)
  - really care about distance to nearest valid event
- 2) Traditional: adjust estimator
  - "edge-corrected" estimator of $\hat{G}(x)$
  - Usual: use the Kaplan-Meier estimator for censored data
  - others have been proposed, avoid reduced sample method
  - but, bias correction increases Var $\hat{G}(x)$
- 3) Radical: adjust expectation
  - Use uncorrected estimator
  - Change theoretical $G(x)$ to account for edge effects
- If goal is to est. $G(x)$, 2) much better
- If goal is to test CSR (or some other process), 3) has higher power
- Estimate theoretical $G(x)$ by simulation

CSR



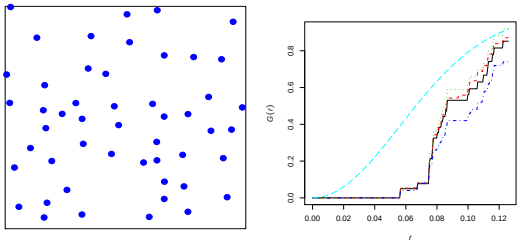
clustering

- More points close: shorter NN distance



inhibitation / regularity

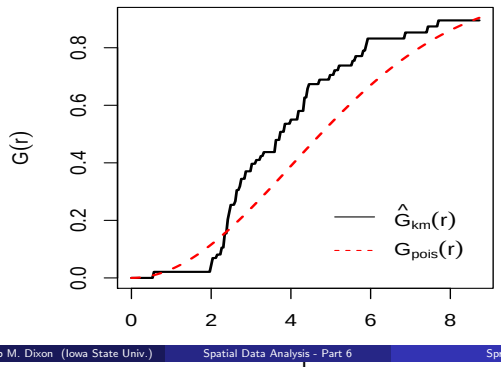
- Fewer points close: longer NN distance
- Hard core process: no points within a minimum distance

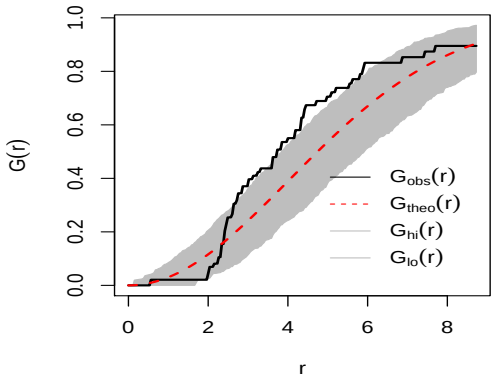


Sampling variance of $\hat{G}(x)$

- How to calculate Var $\hat{G}(x)$?
- Quite a hard problem: 1) edge effects
- 2) "Reflexive NN's": pair of points
  - B is A's NN, A is B's NN
  - same NN distance
  - surprisingly common: P[reflexive] = 0.63 for CSR
  - increases Var $\hat{G}(x)$
- Var $\hat{G}(x)$ has been derived under CSR, ignoring edge effects
- Now, almost always computed by simulation
  - Simulate a realization of null hypothesis process (e.g. CSR)
  - Estimate $\hat{G}(x)$
  - Repeat simulate/estimate 99 or 999 times
  - Calculate Var $G(x)$ at various x
- Or go straight to a confidence interval
  - Calculate0.025 and 0.975 quantiles of $\hat{G}(x)$ at a specified x value
  - repeat for various x's

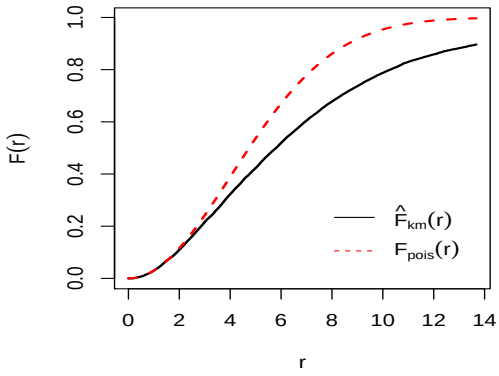
Cypress trees

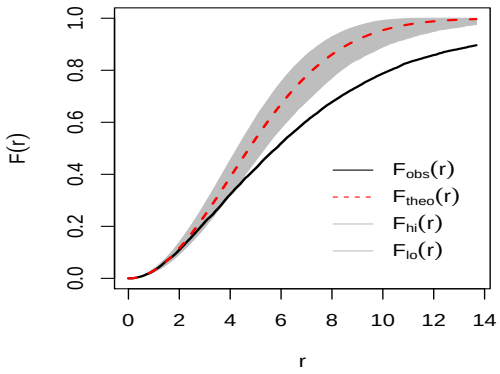




Point-event distance = empty-space function

- cdf of distance from randomly chosen point (not an event) to nearest event
- Usually denoted $F(x)$
- Under CSR, ignoring edge effects: $F(x) = 1 - e^{-\lambda \pi x^2}$
  - same derivation as for $G(x)$
- But now:
  - large distances $\Rightarrow$ clustering, because big areas of empty space
  - small distances $\Rightarrow$ regularity
- Evaluated in same way as $G(x)$
- $F(x)$ more powerful than $G(x)$ to detect clustering
- $G(x)$ more powerful to detect regularity



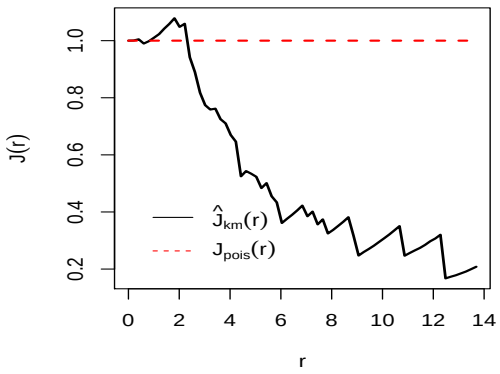


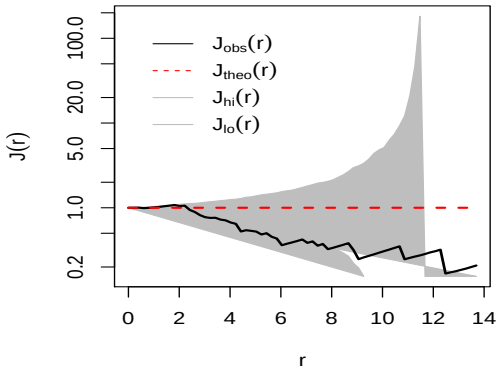
Baddeley's J function

- Can combine $F(x)$ and $G(x)$

$$J(x) = \frac{1 - G(x)}{1 - F(x)}$$

- Interpretation:
  - clustering: J(x) < 1
  - CSR: J(x) $= 1$
  - regularity: J(x) > 1
- Much newer than $F(x)$ or $G(x)$: 1996 paper
- Few have much experience with it





Looking at multiple scales simulataneously

- I've emphasized nearest neighbor (of an event, of a point).
- closest event (to the event, to the point)
- Straightforward extension to 2nd NN (next closest) , 3rd NN, ...
  - Gets harder to interpret
  - and you have a separate plot for each NN rank
- Rethink how to compute the summary
  - Instead of "how far to closest point"
  - think of how many points within a specified distance?
- leads to Ripley's K statistic (Ripley 1976, *J. Appl. Prob*)

Ripley's K function

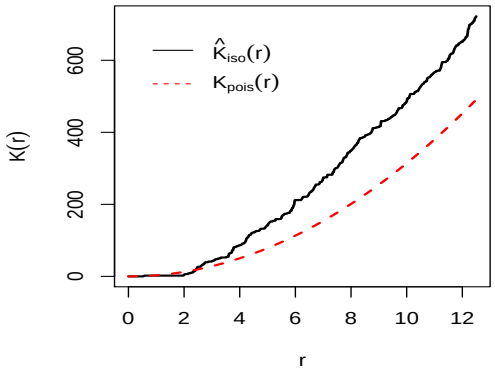
- Like $F(x)$, $G(x)$, and $J(x)$, looks at 2nd order characteristics of a point pattern
- Now, the most commonly used point pattern analysis function
- Provides information at multiple scales simultaneously

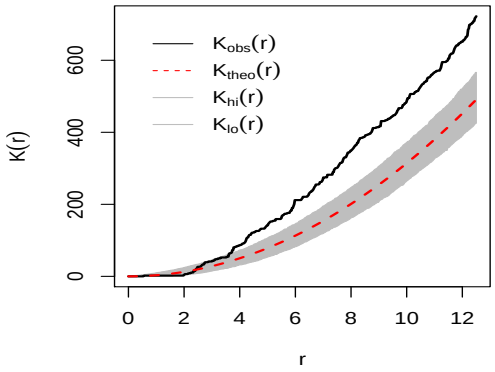
$$K(x) = \frac{1}{\lambda}\text{E (\# events w/i } x \text{ of an event)}$$

- Interpretation, for each distance x:
  - Clustering: $K(x)$ large. Many events close to other events at x or smaller
  - Regularity: $K(x)$ small or 0 at short distances.
- Notes:
  - $K(x)$ can detect clusters of regularly spaced points
  - i.e., different patterns at different scales
  - but it is cumulative (number of points *within* distance x
  - we'll see a refinement, the pair correlation function, that looks at points at distance x
  - which simplifies (greatly) inferring the scale of a pattern

$K(x)$ under CSR

- Expected value, E $\hat{K}(x)$:
  - Under CSR, events are independent, E # in area $A = \lambda A$
  - E $\hat{K}(x)$ = E # in area $\pi x^2/\lambda = \lambda\pi x^2/\lambda = \pi x^2$
- Variance, Var $\hat{K}(x)$:
  - Var # in area $A = \lambda A$
  - so, Var $\hat{K}(x) = \frac{1}{\lambda^2}$ Var # in πx^2
  - $= \pi x^2/\lambda$
  - smaller with more expected points (larger λ)
  - increases with distance, x





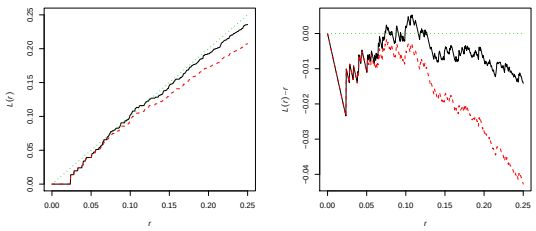
Besag's L function

- $L(x)$ is a variance equalizing transformation
- if $Y \sim \text{Poiss}(X\beta)$, then $\sqrt{Y}$ has constant variance
- Besag's original version

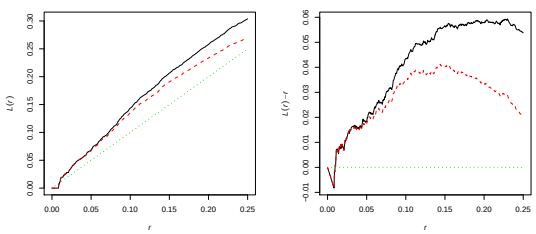
$$L(x) = \sqrt{K(x)/\pi}$$

  - Wiegand and Moloney (2014) call this $L_1(x)$
- Under CSR: L(x) = x, Var $\hat{L}(x)$ approx constant.
- I prefer $L^*(x) = L(x) - x$
  - Wiegand and Moloney (2014) call this $L_2(x)$
- Nice feature of $L^* = L_2(x)$: under CSR, $L^*(x) = 0$
- I believe plots of L^* are much clearer (but you decide which you prefer)

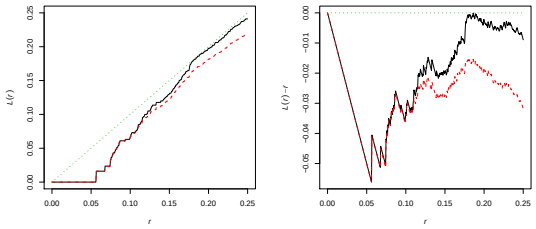
CSR

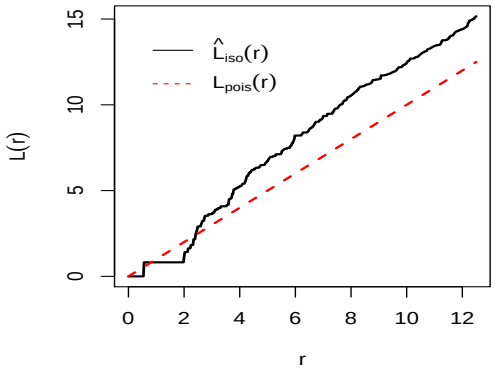


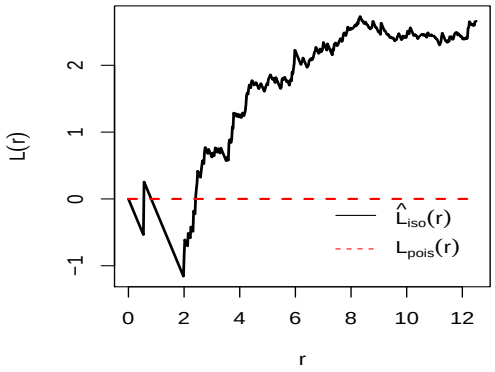
Clustered

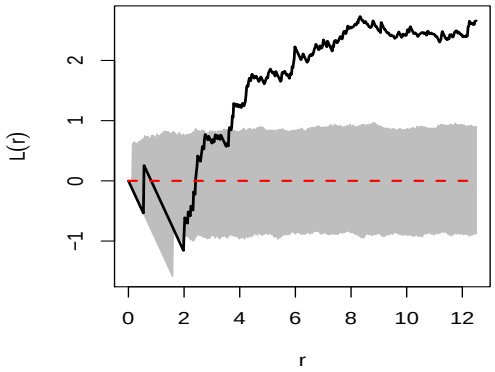


inhibition









Pair correlation functions

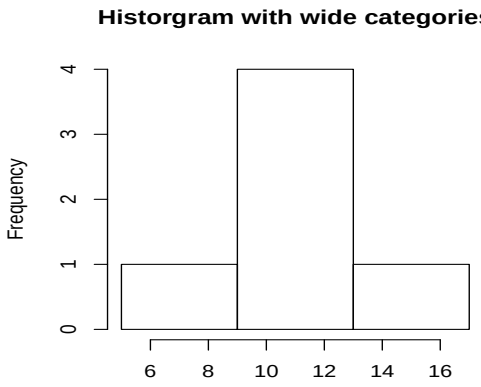
- $K(x)$ and $L(x)$ are cumulative measures
  - Based on number of events *within* x of another event
- What if you want to describe association *at* distance x?
  - Closer to intuition about spatial scale
  - Can untangle multiple processes
    - inhibition at short distances
    - clustering at large distances
- pair-correlation function, $g(x)$ or $\rho(x)$

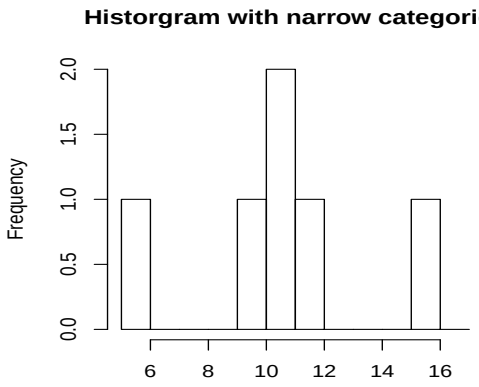
$$g(x) = \frac{1}{2\pi x}\frac{dK(x)}{dx}$$

  - under CSR ($K(x) = \pi x^2$), $\frac{dK(x)}{dx} = 2\pi x$, and $g(x) = 1$
  - $g(x) > 1 \Rightarrow$ events more likely AT distance x than under CSR $\Rightarrow$ clustering at a scale of x
  - $g(x) < 1 \Rightarrow$ events less likely AT distance x than under CSR $\Rightarrow$ repulsion at a scale of x
  - range is $(0, \infty)$ with 1 as the neutral point
  - so often log transform: evaluate $\log g(x)$

Estimators of $g(x)$

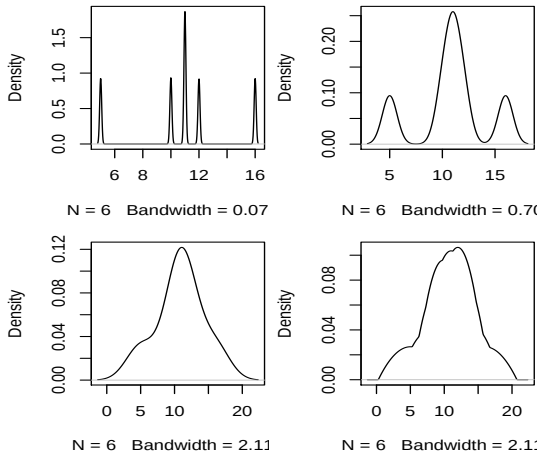
- $g(x)$ is much harder to estimate than $K(x)$
  - $\hat{K}(x)$ is a sum (# events within distance $< x$)
  - $g(x)$ depends on 0/1 variable: is there an event at distance $= x$ or not
  - parallel to the issue that a cdf: $P[X < x]$ is easier to estimate than a pdf: $f[X = x]$
- Two proposed estimators:
  - Wiegand and Moloney "O-ring" estimator: # events within (x, x + dx) equivalent to binning obs. to make a histogram
  - kernel smoothing: much better (both for density estimation and $\hat{g}(x)$)
  - What is the histogram of 5,10,11,11, 12, 16?
    - choice of "bin width" really matters
    - see next two slides



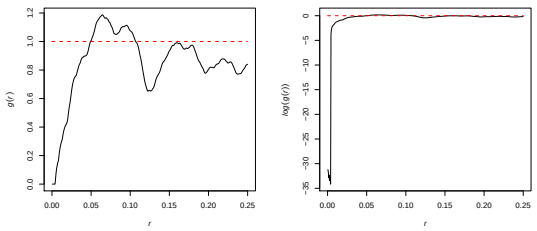


Kernel smoothing

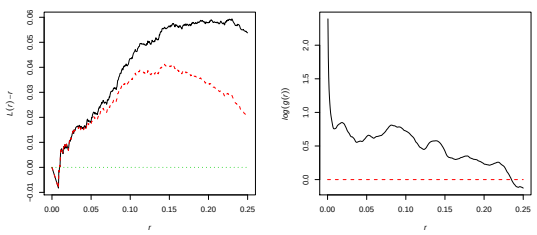
- Histograms estimate probability density for a range of X using only the values in that range
- the variance in the estimated probability depends on the number of obs in the bin
  - wide bin: many points, low variance, but biased estimate (one number for many X values)
  - narrow bin: low bias (small range of X values), but large variance (few obs in bin)
- Density estimation partially avoids this tradeoff and is less dependent on the breaks between categories
- Concept: superimpose little "bumps" of probability around each obs. Add up the probability to estimate $f(x)$
- result depends on sd of each "bump"
- sd called "bandwidth"
- result also depends on "kernel", i.e. the shape of the bump
- notice that range of density estimate is wider than data range.



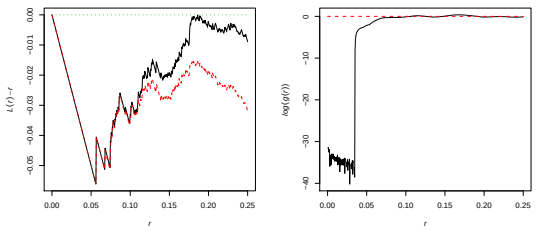
CSR

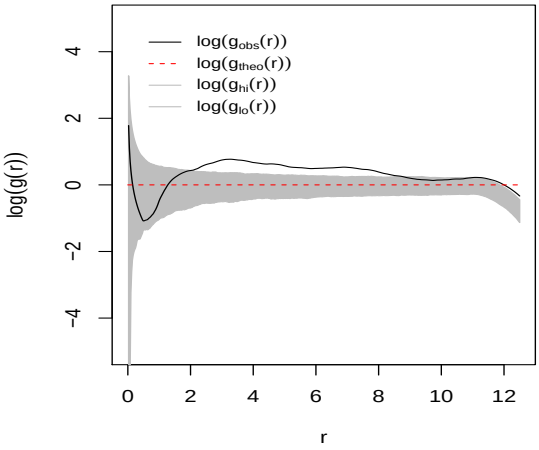


Clustered



inhibition





Interpreting / using measures

- 1) to test CSR: pointwise tests
  - estimate $\hat{L}(x)$ at a range of distances, x
  - use simulation to calculate point-wise quantiles of $\hat{L}(x)$
  - plot $\hat{L}(x)$ and simulation envelope
  - interpret deviations above and below expected
  - consider distance x_1, then distance x_2
  - called pointwise-tests.
  - Type I error rate, α level, correct for one test
  - One issue (serious): multiple testing
    - doing many tests, one at each distance
    - P[reject — CSR at any distance] is much larger than P[reject — CSR]
    - especially for cumulative summary functions, $K(x)$ and $L(x)$
    - Quite hard to do a true level α test
    - usual approaches don't work well because $\hat{L}(x_1)$ and $\hat{L}(x_2)$ are correlated

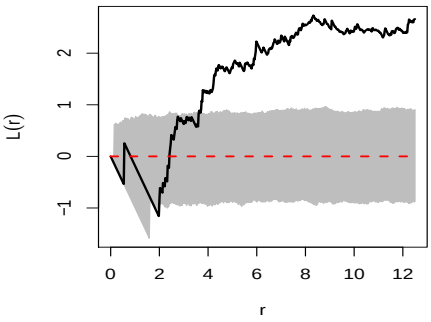
Interpreting / using measures

- 2) summary tests of CSR
  - Calculate a summary statistic across "relevant" range of distances
  - Two common choices, using $L(x)$ as example:
    - $s = \max_x[\mid \hat{L}(x) - L(x) \mid]$ (maximum statistic, Maximum Absolute Deviation)
    - $s = \int_x [\hat{L}(x) - L(x)]^2$ (integral statistic, Loosmore and Ford test)
    - Both computed by evaluating "interesting" set of x, finding max or sum
  - $L(x)$ can be theoretical expectation ($K(x) = \pi x^2$, $L_2(x) = 0$)
  - Or, $L(x)$ computed as average of n simulations (see below)
    - accounts for bias due to edge corrections
  - Integral better when consistent but small deviations above expected curve
    - more commonly used
  - Max better when large excursion from theoretical value for a small range of distances

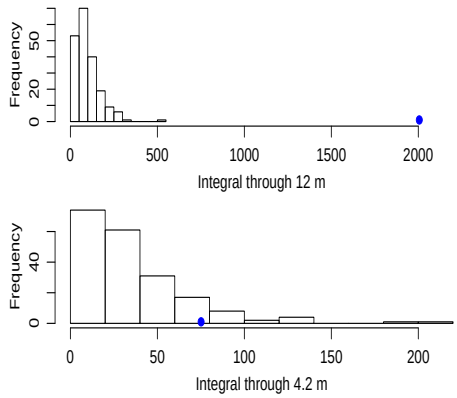
Summary tests

- Turning into a hypothesis test
  - Have s_{obs} from the observed pattern
  - Simulate many (39, 99, 999) random patterns under H0 (e.g., CSR)
  - Calculate summary statistic for observed data and each simulated data set
  - Calculate P[as or more extreme summary statistic] = p-value
  - Usually one-sided definition of more extreme (only care about large s)
- This avoids multiple testing issues and gives valid p-value

Summary tests: Cypress pointwise tests

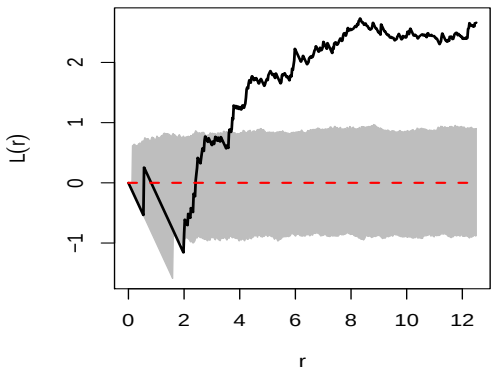


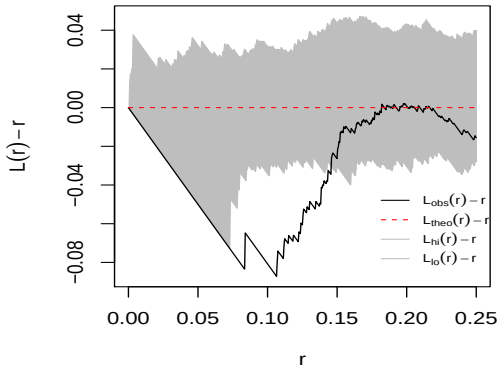
Summary tests: Cypress summary tests



Summary tests

- Need to choose upper and lower distances
  - best when chosen to be biologically relevant.
  - DON'T look for the most significant region
- most commonly used to test CSR
  - But you specify the null hypothesis
  - same approach can be used for any point process model (examples coming soon)
- The integral and especially the maximum statistic assume Var $\hat{L}(x)$ approximately constant
  - Don't use $K(x)$: because Var $K(x)$ is definitely not constant
  - Use $L(x)$ instead, approximately constant variance
  - But not perfect (see next two plots)
  - Can transform $G(x)$ or $F(x)$ (both proportions), e.g. $\sin^{-1}\sqrt{\hat{G(x)}}$
- There are studentized summary statistics, if unequal Var is bad





A poor use of a test of CSR

- Until a few years ago, it was fashionable to
  - map locations of all things in an area. Usually trees or other plants, could be animal nests
  - usually many species
  - for each species, test CSR (usually using K/L functions)
  - tabulate # species that are clustered, # random, # regular
- then make ecological conclusions about the community
- do you see the issue here?

A poor use of a test of CSR

- If you don't reject H0: CSR for a species, do you know that species is randomly distributed?
- What if you only had 10 individuals for that species?
  - Statistical power to detect 'not-CSR' is really small
- failure to reject H0 does not $\Rightarrow$ H0 is true
- In my experience (mostly with trees)
  - large # events: detect clustering, sometimes regularity
  - small # events: accept H0
- If you expect intensity to vary over a study area, that introduces clustering.
- If you believe that non-random spatial patterns are the norm, the hypothesis test is really telling you only whether you have a sufficiently large sample size to detect that non-random pattern.

Interpreting / using measures

- 3) Estimating # "excess events"
  - if pattern is CSR, expect $\lambda\pi x^2$ events within radius x of another event
  - data says an average of $\lambda\hat{K}(x)$ events within radius x of another event
  - $\lambda\hat{K}(x) - \lambda\pi x^2$ is average "excess events"
  - describes magnitude of clustering in subject-matter terms
  - less frequently used is $\frac{\hat{K}(x)}{\pi x^2} - 1$
    proportion of excess events at distance x
- Cypress tree illustration
  - $\hat{\lambda} = 98/(50 \times 200) = 0.0098$
  - at distance of 10m, $\hat{K}(10) = 488.6 \Rightarrow$ ave. of 4.7 cypress trees within 10m of another cypress tree
  - $\pi 10^2 = 314.7 \Rightarrow$ 1.7 excess cypress trees within 10m of another.
  - Or, $\frac{488.6}{314.7} - 1 = 0.55 \Rightarrow$ 55% more cypress trees within 10m of another.

Interpreting / using measures

- 4) Describing spatial scale
  - "scale" is a tricky concept. Various definitions
  - Here, scale = distance(s) at which events repulse each other or attract each other
  - A distance-specific concept
  - Many studies have used $\hat{K}(x)$ or $\hat{L}(x)$ to estimate scale, e.g. find x where $L(x)$ is most different from theoretical value
- Increasingly understood to be wrong
  - Both $\hat{K}(x)$ and $\hat{L}(x)$ are cumulative functions: # points within circle of radius x
  - Small # at distance x may be because repulsion (fewer pts.) at distances $< x$, even if strong clustering at x
  - Really want to know what is going on AT distance x, not $\leq x$
- Use pair-correlation function

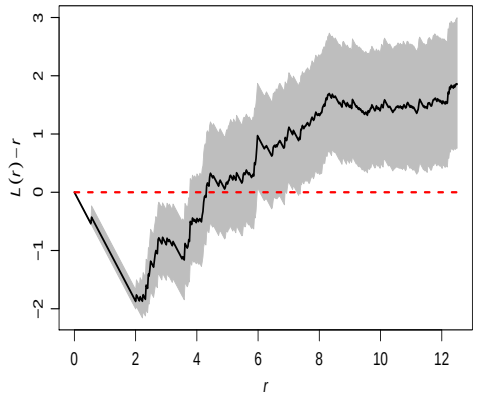
Interpreting / using measures

- 5) How precise is $\hat{L}(x)$ or $\hat{g}(x)$?
- Not the width of the Null hypothesis envelopes
- Precision of $\hat{L}(x)$ or $\hat{g}(x)$
  - Certainly depends on $N = \#$ points
  - But also on the spatial pattern
    - $\hat{K}(x)$ more variable for clustered patterns
- If you know the true spatial pattern, simulate from that pattern and calculate envelope
- If you don't know the true pattern, use a bootstrap
- Point pattern bootstrap proposed by Loh, 2008

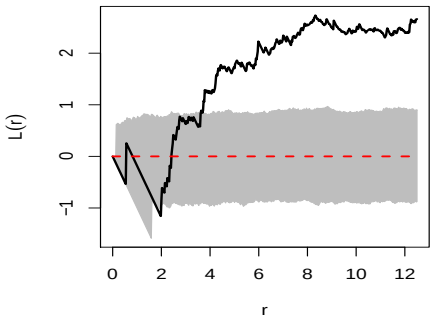
The bootstrap

- General method for estimating precision of a statistic
- Uses resampling the data to approximate the unknown sampling distribution of a statistic
- Gives you the se of a statistic or a confidence interval for a statistic
- CI much more common
- Not the same as a randomization test or a null hypothesis test
  - Hypothesis test: simulate / resample assuming H0 (CSR, no diff. in means)
  - Bootstrap: simulate / resample assuming Ha (arbitrary pattern, non-zero diff)
- Extremely useful tool for "difficult" problems
- Usual forms of bootstrap don't work for point pattern data
- Problem is that one point contributes to many $L(x)$
- Loh devised something that (so far) is acceptable sometimes
  - resample contributions to $\hat{g}(x)$ or $\hat{L}(x)$
- Issues when bootstrap average curve not same as data curve (see below)

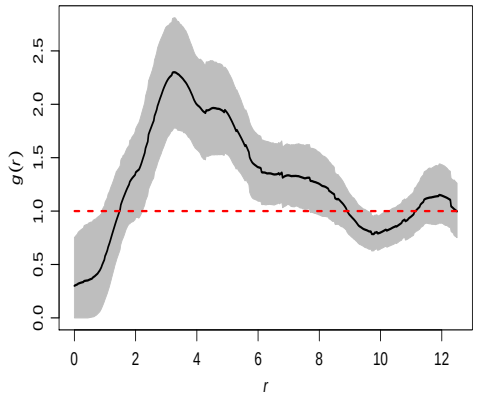
Cypress L(x) bootstrap



Cypress L(x) CSR (null) envelope



Cypress g(x) bootstrap



Cypress g(x) CSR (null) envelope